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The basic contrasts of a block experimental design with special reference to the notion of general balance

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Summary

A unified theory of basic contrasts of a block design is presented and, in relation to it, the notions of orthogonal block structure and of general balance are recalled. Under the randomization model it is shown that these two notions are applicable to proper block designs only. In particular the role of the basic contrasts in defining the general balance of a block design is indicated, and the practical meaning of the balance with respect to these contrasts is discussed.

1. Introduction

The concept of basic contrasts has been introduced by Pearce, Caliński and Marshall (1974) and again reported in the book by Pearce (1983, Section 3.6). Meanwhile the concept of general balance, originating from the work of Nelder (1965a, b), has been formalized by Houtman and Speed (1983). The latter authors have recognized an important relation between the basic contrasts and the notion of general balance (Houtman and Speed, 1983, p.1072). The purpose of this paper is to present a unified theory of the basic contrasts of a block design and to show how it is related to the notion of general balance of the design. This will be done with reference to the randomization model recently recalled and examined for block experiments by Caliński and Kageyama (1991).

The randomization model is restated in Section 2 and the concept of basic contrasts is recalled in Section 3, together with their intra-block analysis. Further results concerning basic contrasts are confined here to proper block designs (of

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equal block sizes). This is due to the fact that only for these designs the notions of orthogonal block structure (Section 4) and of general balance (Section 5) are applicable under the randomization model. Examples illustrating the meaning of general balance are given in Section 5 and relevant remarks in Section 6.

The notation in this paper will be as that used by Caliński and Kageyama (1991), unless otherwise stated.

2. The randomization model and its submodels

Suppose that units of a block experiment are randomized before entering the experiment in the way described by Nelder (1954), by randomly permuting blocks within a total area of them and by randomly permuting units within the blocks. Then, assuming the usual unit-treatment additivity, and also assuming, as usual, that the technical errors are uncorrelated, with zero expectation and a constant variance, independent of the treatments in particular, the model of the variables observed on the n units actually used in the experiment can be written in matrix notation as

$$\mathbf{y} = \boldsymbol{\Delta}'\boldsymbol{\tau} + \mathbf{D}'\boldsymbol{\beta} + \boldsymbol{\eta} + \mathbf{e}\,, \tag{2.1}$$

where $\mathbf{y}$ is a column vector of n observed variables, $\boldsymbol{\Delta}'$ is an $n{\times}v$ design matrix for treatments, v being the number of treatments compared in the experiment, and $\mathbf{D}'$ is an $n{\times}b$ design matrix for blocks, b being the number of blocks used in the experiment, and where, accordingly, $\boldsymbol{\tau}$ is a vector of treatment parameters $\{\tau_i\}$, $\boldsymbol{\beta}$ is a vector of block random effects $\{\beta_j\}$, $\boldsymbol{\eta}$ is a vector of unit errors $\{\eta_{l(j)}\}$ and $\mathbf{e}$ is a vector of technical errors $\{e_{l(j)}\}$, $l(j)$ denoting the unit l in block j. The expectation vector and the dispersion matrix (covariance matrix) derived for the model (2.1) from the assumptions and the randomizations involved are

$$\mathrm{E}(\mathbf{y}) = \boldsymbol{\Delta}'\boldsymbol{\tau} \tag{2.2}$$

and

$$\mathrm{Cov}(\mathbf{y}) = (\mathbf{D}'\mathbf{D} - N_B^{-1}\mathbf{1}_n\mathbf{1}_n')\,\sigma_B^2 + (\mathbf{I}_n - K_H^{-1}\mathbf{D}'\mathbf{D})\sigma_U^2 + \mathbf{I}_n\sigma_e^2\,, \tag{2.3}$$

respectively, where $\mathbf{1}_n$ is an $n{\times}1$ vector of units and $\mathbf{I}_n$ is an identity matrix of order n, N_B is the potential number of blocks from which b have been randomly chosen for the experiment ($b \le N_B$), K_H is a weighted harmonic average of the potential (available) numbers of units within the blocks, from which units in numbers $\{k_j\}$ have been chosen for the experiment after the randomization, and where the variance components are defined according to the relations

$$\mathrm{Cov}(\beta_j, \beta_{j'}) = \begin{cases} N_B^{-1}(N_B - 1)\sigma_B^2 & \text{if } j = j' , \\ -N_B^{-1}\sigma_B^2 & \text{if } j \neq j' , \end{cases}$$

and

$$\mathrm{Cov}(\eta_{l(j)}, \eta_{l'(j')}) = \begin{cases} K_H^{-1}(K_H - 1)\sigma_U^2 & \text{if } j = j' \text{ and } l = l' , \\ -K_H^{-1}\sigma_U^2 & \text{if } j = j' \text{ and } l \neq l' , \\ 0 & \text{if } j \neq j' . \end{cases}$$

(For detailed derivations see Caliński and Kageyama, 1988.)

As shown by Caliński and Kageyama (1991, Section 2.2), under the model (2.1) the best linear unbiased estimators (BLUEs) of linear treatment parametric functions exist in very restrictive circumstances only. More precisely, for a function $\mathbf{c}'\boldsymbol{\tau} = \mathbf{s}'\mathbf{r}^{\delta}\boldsymbol{\tau}$, where $\mathbf{r}^{\delta} = \boldsymbol{\Delta}\boldsymbol{\Delta}'$, the BLUE exists if and only if the vector $\mathbf{s}$ is related to the incidence matrix $\mathbf{N} = \boldsymbol{\Delta}\mathbf{D}'$ of the design by the condition (a) $\mathbf{N}'\mathbf{s} = \mathbf{0}$ or (b) $\mathbf{N}'\mathbf{s} \neq \mathbf{0}$ with the elements of $\mathbf{N}'\mathbf{s}$ all equal if the design is connected, and equal within any connected subdesign otherwise.

If the above conditions are not satisfied for parametric functions of interest then the usual procedure is to resolve the model (2.1) into three submodels (two for contrasts), in accordance with the stratification of the experimental units. This can be represented by the decomposition

$$\mathbf{y} = \mathbf{y}_1 + \mathbf{y}_2 + \mathbf{y}_3 , \tag{2.4}$$

resulting from orthogonal projections of $\mathbf{y}$ on subspaces related to the three "strata":

1st – of units within blocks, the "intra-block" stratum,

2nd – of blocks within the total area, the "inter-block" stratum,

3rd – of the total area

(using the terminology of Pearce, 1983, p.109). Explicitly (see Caliński and Kageyama, 1991, Section 3),

$$\mathbf{y}_1 = \boldsymbol{\varphi}_1\mathbf{y} , \quad \mathbf{y}_2 = \boldsymbol{\varphi}_2\mathbf{y} \text{ and } \mathbf{y}_3 = \boldsymbol{\varphi}_3\mathbf{y} , \tag{2.5}$$

where the projectors $\boldsymbol{\varphi}_\alpha$, $\alpha = 1,2,3$, are defined, with $\mathbf{k}^{-\delta} = (\mathbf{D}\mathbf{D}')^{-1}$, as

$$\boldsymbol{\varphi}_1 = \mathbf{I} - \mathbf{D}'\mathbf{k}^{-\delta}\mathbf{D} , \quad \boldsymbol{\varphi}_2 = \mathbf{D}'\mathbf{k}^{-\delta}\mathbf{D} - n^{-1}\mathbf{1}_n\mathbf{1}_n' , \quad \text{and} \quad \boldsymbol{\varphi}_3 = n^{-1}\mathbf{1}_n\mathbf{1}_n' .$$

They satisfy the conditions

$$\boldsymbol{\varphi}_\alpha = \boldsymbol{\varphi}_\alpha' , \quad \boldsymbol{\varphi}_\alpha\boldsymbol{\varphi}_\alpha = \boldsymbol{\varphi}_\alpha , \quad \boldsymbol{\varphi}_\alpha\boldsymbol{\varphi}_{\alpha'} = \mathbf{0} \text{ for } \alpha \neq \alpha' , \quad \boldsymbol{\varphi}_1 + \boldsymbol{\varphi}_2 + \boldsymbol{\varphi}_3 = \mathbf{I}_n , \tag{2.6}$$

also

$$\boldsymbol{\varphi}_1\mathbf{D}' = \mathbf{0} \text{ and } \boldsymbol{\varphi}_\alpha\mathbf{1}_n = \mathbf{0} \text{ for } \alpha = 1,2 . \tag{2.7}$$

The submodels (2.5), called "intra-block", "inter-block" and "total-area", respectively, have the following properties:

$$E(\mathbf{y}_1) = \boldsymbol{\varphi}_1\boldsymbol{\Delta}'\boldsymbol{\tau}\ ,\ \mathrm{Cov}(\mathbf{y}_1) = \boldsymbol{\varphi}_1(\sigma_U^2 + \sigma_e^2)\ , \tag{2.8}$$

$$E(\mathbf{y}_2) = \boldsymbol{\varphi}_2\boldsymbol{\Delta}'\boldsymbol{\tau}\ ,\ \mathrm{Cov}(\mathbf{y}_2) = \boldsymbol{\varphi}_2\mathbf{D}'\mathbf{D}\boldsymbol{\varphi}_2(\sigma_B^2 - K_H^{-1}\sigma_U^2) + \boldsymbol{\varphi}_2(\sigma_U^2 + \sigma_e^2)\ , \tag{2.9}$$

$$E(\mathbf{y}_3) = \boldsymbol{\varphi}_3\boldsymbol{\Delta}'\boldsymbol{\tau}\ ,\ \mathrm{Cov}(\mathbf{y}_3) = \boldsymbol{\varphi}_3[(n^{-1}\mathbf{k}'\mathbf{k} - N_B^{-1}n)\sigma_B^2 + (1 - K_n^{-1}n^{-1}\mathbf{k}'\mathbf{k})\,\sigma_U^2 + \sigma_e^2]\ , \tag{2.10}$$

where $\mathbf{k} = \mathbf{N}'\mathbf{1}_v$ is the vector of block sizes of the design. Evidently, if the block sizes are all equal, i.e. the design is proper, then the dispersion matrices of the inter-block and total-area submodels are simplified. However, the properties of the intra-block submodel remain the same, whether the design is proper or not. This means that the intra-block analysis (as that based on the intra-block submodel) can be considered generally, for any block design.

3. Basic contrasts and their intra-block analysis

In Section 4 of the paper by Caliński and Kageyama (1991) it has been shown that certain contrasts of treatment parameters play a special role in the analysis of block designs. A relation between the design and these contrasts is given in terms of eigenvectors and corresponding eigenvalues of the C-matrix ($\mathbf{C} = \boldsymbol{\Delta}\boldsymbol{\varphi}_1\boldsymbol{\Delta}'$) of the design with respect to $\mathbf{r}^\delta$, the diagonal matrix of treatment replications. The eigenvectors represent the contrasts, and the eigenvalues express the efficiency factors of the design for the intra-block estimation of the contrasts. Thus, to each block design there corresponds a set of contrasts, not necessarily unique, for which the efficiencies of estimation in the intra-block analysis are readily defined. This relation was originally noticed by Jones (1959). Algebraically, this characterization is supplied by the spectral decomposition

$$\mathbf{C} = \mathbf{r}^\delta\Big(\sum_{i=1}^{h} \varepsilon_i \mathbf{s}_i \mathbf{s}_i'\Big)\mathbf{r}^\delta\ ,$$

where $\{\varepsilon_i\}$ are the eigenvalues and $\{\mathbf{s}_i\}$ are the corresponding $\mathbf{r}^\delta$-orthonormal eigenvectors of $\mathbf{C}$, with respect to $\mathbf{r}^\delta$, and where h = rank($\mathbf{C}$). Contrasts represented by the vectors $\{\mathbf{s}_i\}$ have been termed by Pearce et al. (1974) as follows.

Definition 3.1. For any block design, contrasts $\{\mathbf{c}_i'\boldsymbol{\tau} = \mathbf{s}_i'\mathbf{r}^\delta\boldsymbol{\tau}\ ,\ i=1,2,\dots,v-1\}$ are said to be basic contrasts of the design if the vectors $\{\mathbf{s}_i\}$ are $\mathbf{r}^\delta$-orthonormal eigenvectors of the matrix $\mathbf{C} = \boldsymbol{\Delta}\boldsymbol{\varphi}_1\boldsymbol{\Delta}' = \mathbf{r}^\delta - \mathbf{N}\mathbf{k}^{-\delta}\mathbf{N}'$ of the design with respect to $\mathbf{r}^\delta$.

Note that the eigenvalues $\{\varepsilon_i\}$ of $\mathbf{C}$ with respect to $\mathbf{r}^\delta$ are then the efficiency factors of the design for estimating the corresponding basic contrasts in the intra-block analysis. In this section, when the term "efficiency factor for a contrast" is used, it should be understood as the efficiency factor of the design for estimating the contrast in the intra-block analysis.

The following results give sense to the term "basic" used in Definition 3.1.

Theorem 3.1. Let $\{\mathbf{c}_i'\boldsymbol{\tau} = \mathbf{s}_i'\mathbf{r}^\delta\boldsymbol{\tau},\ i=1,2,...,v-1\}$ be any set of basic contrasts of a block design and let $\{\varepsilon_i,\ i=1,2,...,v-1\}$ be the corresponding efficiency factors. Then

(i) the intra-block analysis provides the BLUEs, of the form

$$(\widehat{\mathbf{c}_i'\boldsymbol{\tau}})_{\text{intra}} = \varepsilon_i^{-1}\mathbf{s}_i'\mathbf{Q}_1 = \varepsilon_i^{-1}\mathbf{c}_i'\mathbf{r}^{-\delta}\mathbf{Q}_1 \quad (\mathbf{Q}_1 = \boldsymbol{\Delta}\boldsymbol{\varphi}_1\mathbf{y}) \tag{3.1}$$

with the variances

$$\text{Var}[(\widehat{\mathbf{c}_i'\boldsymbol{\tau}})_{\text{intra}}] = \varepsilon_1^{-1}\sigma_1^2 \qquad (\sigma_1^2 = \sigma_U^2 + \sigma_e^2) \tag{3.2}$$

and the covariances

$$\text{Cov}[(\widehat{\mathbf{c}_i'\boldsymbol{\tau}})_{\text{intra}},\ (\widehat{\mathbf{c}_{i'}'\boldsymbol{\tau}})_{\text{intra}}] = 0 \qquad (i \neq i'), \tag{3.3}$$

for those of the basic contrasts for which the efficiency factors are nonzero (positive), and

(ii) there are no BLUEs in the intra-block analysis for those basic contrasts for which the efficiency factors are zero.

Proof. Part (i) follows from Theorem 4.1 of Caliński and Kageyama (1991), formulae (3.1) and (3.2) following from (4.8) and (4.9) there, respectively, while (3.3) follows from the formula

$$\text{Cov}(\mathbf{Q}_1) = \boldsymbol{\Delta}\boldsymbol{\varphi}_1\boldsymbol{\Delta}'\sigma_1^2, \tag{3.4}$$

holding on account of (2.8), and from the fact that $\mathbf{s}_i'\boldsymbol{\Delta}\boldsymbol{\varphi}_1\boldsymbol{\Delta}'\mathbf{s}_{i'} = 0$ if $i \neq i'$, due to the properties of $\{\mathbf{s}_i\}$ given in Definition 3.1. Part (ii) is obvious, as $\boldsymbol{\Delta}\boldsymbol{\varphi}_1\boldsymbol{\Delta}'\mathbf{s} = \mathbf{0}$ implies that $\mathbf{s}'\mathbf{Q}_1 = 0$. □

Theorem 3.2. For any block design for which the vectors $\mathbf{s}_1, \mathbf{s}_2, ..., \mathbf{s}_h$ represent basic contrasts receiving nonzero (positive) efficiency factors, a set of contrasts $\mathbf{U}'\boldsymbol{\tau}$ obtains the BLUEs in the intra-block analysis if and only if the matrix $\mathbf{U}$ can be written as $\mathbf{U} = \mathbf{r}^\delta\mathbf{S}\mathbf{A}$, where $\mathbf{S} = [\mathbf{s}_1 : \mathbf{s}_2 : ... : \mathbf{s}_h]$, and $\mathbf{A} = [\mathbf{a}_1 : \mathbf{a}_2 : ... : \mathbf{a}_h]'$ is some matrix of h rows. If $\mathbf{U}$ is such, then the BLUEs provided by the intra-block analysis are of the form

$$(\widehat{\mathbf{U}'\boldsymbol{\tau}})_{\text{intra}} = \mathbf{A}'\boldsymbol{\varepsilon}^{-\delta}\mathbf{S}'\mathbf{Q}_1 = \sum_{i=1}^{h} \varepsilon_i^{-1}\mathbf{a}_i\mathbf{s}_i'\mathbf{Q}_1 \quad (\mathbf{Q}_1 = \boldsymbol{\Delta}\boldsymbol{\varphi}_1\mathbf{y}) \tag{3.5}$$

and their dispersion matrix is of the form

$$\text{Cov}[(\widehat{\mathbf{U}'\boldsymbol{\tau}})_{\text{intra}}] = \mathbf{A}'\boldsymbol{\varepsilon}^{-\delta}\mathbf{A}\sigma_1^2 = \sum_{i=1}^{h} \varepsilon_i^{-1}\mathbf{a}_i\mathbf{a}_i'\,\sigma_1^2 \quad (\sigma_1^2 = \sigma_U^2 + \sigma_e^2)\ , \tag{3.6}$$

where $\boldsymbol{\varepsilon}^{\delta} = \text{diag}[\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_h]$ and $\boldsymbol{\varepsilon}^{-\delta} = (\boldsymbol{\varepsilon}^{\delta})^{-1}$.

Proof. On account of Theorem 3.1 of Caliński and Kageyama (1991), the intra-block analysis provides the BLUEs for $\mathbf{U}'\boldsymbol{\tau}$ if and only if the columns of the matrix $\mathbf{U}$ are linear combinations of the matrix $\boldsymbol{\Delta}\boldsymbol{\varphi}_1\boldsymbol{\Delta}'$, i.e. if and only if there is a matrix $\mathbf{A}^*$ such that $\mathbf{U} = \boldsymbol{\Delta}\boldsymbol{\varphi}_1\boldsymbol{\Delta}'\mathbf{A}^*$. But, since

$$\mathbf{C} = \boldsymbol{\Delta}\boldsymbol{\varphi}_1\boldsymbol{\Delta}' = \mathbf{r}^{\delta}\mathbf{S}\boldsymbol{\varepsilon}^{\delta}\mathbf{S}'\mathbf{r}^{\delta}\ , \tag{3.7}$$

the condition for $\mathbf{U}$ can be written as

$$\mathbf{U} = \mathbf{r}^{\delta}\mathbf{S}\boldsymbol{\varepsilon}^{\delta}\mathbf{S}'\mathbf{r}^{\delta}\mathbf{A}^* = \mathbf{r}^{\delta}\mathbf{S}\mathbf{A}\ ,$$

with $\mathbf{A} = \boldsymbol{\varepsilon}^{\delta}\mathbf{S}'\mathbf{r}^{\delta}\mathbf{A}^*$. On the other hand, if $\mathbf{U} = \mathbf{r}^{\delta}\mathbf{S}\mathbf{A}$, then one can write

$$\mathbf{U} = \mathbf{r}^{\delta}\mathbf{S}\boldsymbol{\varepsilon}^{\delta}\mathbf{S}'\mathbf{r}^{\delta}\mathbf{S}\boldsymbol{\varepsilon}^{-\delta}\mathbf{A} \quad (\text{since } \mathbf{S}'\mathbf{r}^{\delta}\mathbf{S} = \mathbf{I}_h)$$
$$= \boldsymbol{\Delta}\boldsymbol{\varphi}_1\boldsymbol{\Delta}'\mathbf{A}^*, \text{ with } \mathbf{A}^* = \mathbf{S}\boldsymbol{\varepsilon}^{-\delta}\mathbf{A}\ .$$

This proves the first part of the theorem. To prove the second, note that since $\mathbf{U} = \boldsymbol{\Delta}\boldsymbol{\varphi}_1\boldsymbol{\Delta}'\mathbf{S}\boldsymbol{\varepsilon}^{-\delta}\mathbf{A}$, the functions $\mathbf{A}'\boldsymbol{\varepsilon}^{-\delta}\mathbf{S}'\boldsymbol{\Delta}\mathbf{y}_1 = \mathbf{A}'\boldsymbol{\varepsilon}^{-\delta}\mathbf{S}'\mathbf{Q}_1$ are, again on account of Theorem 3.1 of Caliński and Kageyama (1991), the BLUEs of the contrasts $\mathbf{U}'\boldsymbol{\tau}$. (That these are contrasts follows from the equality $\mathbf{1}_v'\boldsymbol{\Delta}\boldsymbol{\varphi}_1 = \mathbf{0}$.) Thus, (3.5) is established, while (3.6) follows from (3.4) and (3.7). □

Corollary 3.1. For any block design for which the vectors $\mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_\rho$ represent basic contrasts receiving the unit efficiency factors, a set of contrasts $\mathbf{U}_0'\boldsymbol{\tau} = \mathbf{A}_0'\mathbf{S}_0'\mathbf{r}^{\delta}\boldsymbol{\tau}$, where $\mathbf{S}_0 = [\mathbf{s}_1 : \mathbf{s}_2 : \ldots : \mathbf{s}_\rho]$ and $\mathbf{A}_0 = [\mathbf{a}_1 : \mathbf{a}_2 : \ldots : \mathbf{a}_\rho]'$ is some matrix of ρ rows, obtains the BLUEs under the overall model (2.1), in the form

$$\widehat{\mathbf{U}_0'\boldsymbol{\tau}} = \mathbf{A}_0'\mathbf{S}_0'\boldsymbol{\Delta}\mathbf{y} = \sum_{i=1}^{\rho} \mathbf{a}_i\mathbf{s}_i'\boldsymbol{\Delta}\mathbf{y}\ , \tag{3.8}$$

with the dispersion matrix

$$\operatorname{Cov}(\widehat{\mathbf{U}'_0\boldsymbol{\tau}}) = \mathbf{A}'_0\mathbf{A}_0\,\sigma_1^2 = \sum_{i=1}^{\rho} \mathbf{a}_i\mathbf{a}'_i\,\sigma_1^2 \quad (\sigma_1^2 = \sigma_U^2 + \sigma_e^2)\ . \tag{3.9}$$

Proof. This result follows from Theorem 3.2 above and Corollary 2.1(a) of Caliński and Kageyama (1991).

Remark 3.1. (a) In the notation of Corollary 3.1, a block design for which $\rho \geq 1$ can be called orthogonal for the set of contrasts $\mathbf{U}'_0\boldsymbol{\tau} = \mathbf{A}'_0\mathbf{S}'_0\mathbf{r}^{\delta}\boldsymbol{\tau}$.

(b) It follows from Theorem 3.2 and Corollary 3.1 that the efficiency factor of a block design for a contrast $\mathbf{u}'\boldsymbol{\tau} = \mathbf{a}'\mathbf{S}'\mathbf{r}^{\delta}\boldsymbol{\tau}$ estimated in the intra-block analysis is of the form

$$\varepsilon[(\widehat{\mathbf{u}'\boldsymbol{\tau}})_{\text{intra}}] = \mathbf{a}'\mathbf{a}\,/\,\mathbf{a}'\boldsymbol{\varepsilon}^{-\delta}\mathbf{a}$$

(see also Caliński, Ceranka and Mejza, 1980, p.60).

From the results presented above , it follows that three main subsets of basic contrasts can be distinguished. The first contains contrasts represented by eigenvectors $\mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_\rho$ corresponding to eigenvalues equal to 1. They all are estimated with full efficiency in the intra-block analysis, their BLUEs under the intra-block submodel being simultaneously the BLUEs under the overall model (2.1). The second subset is that of contrasts represented by eigenvectors $\mathbf{s}_{\rho+1}, \ldots, \mathbf{s}_h$ corresponding to positive eigenvalues less than 1 (in nonincreasing order). For all of them the intra-block analysis provides the BLUEs with efficiencies less than 1, due to partially confounding the contrasts with blocks. Finally, the third subset contains contrasts represented by $\mathbf{s}_{h+1}, \ldots, \mathbf{s}_{v-1}$ corresponding to eigenvalues equal to 0. The intra-block analysis provides BLUEs for none of them, due to totally confounding these contrasts with blocks. In a connected design the last subset is empty.

As to the second subset of basic contrasts, it might be worth noting that although the $h-\rho$ efficiency factors for contrasts belonging to this subset may all be different, the number of their distinct values will usually be smaller than $h-\rho$. This gives rise to the following notation.

The spectral decomposition (3.7) can be written in the form

$$\mathbf{C} = \boldsymbol{\Delta}\boldsymbol{\varphi}_1\boldsymbol{\Delta}' = \mathbf{r}^{\delta}\sum_{\beta=0}^{m-1} \varepsilon_\beta \mathbf{H}_\beta \mathbf{r}^{\delta}\ , \tag{3.10}$$

where

$$\mathbf{H}_\beta = \sum_{j=1}^{\rho_\beta} \mathbf{s}_{\beta j}\mathbf{s}'_{\beta j}\ ,$$

$\mathbf{s}_{\beta 1}, \mathbf{s}_{\beta 2}, ..., \mathbf{s}_{\beta \rho_\beta}$ being $\mathbf{r}^\delta$-orthonormal eigenvectors of $\mathbf{C}$ with respect to $\mathbf{r}^\delta$, corresponding to a common eigenvalue ε_β of multiplicity ρ_β (with $\rho_0 \equiv \rho$), and where $m-1$ is the number of distinct, less than 1, nonzero (positive) eigenvalues of $\mathbf{C}$ with respect to $\mathbf{r}^\delta$.

With regard to (3.10) the following result is useful.

Corollary 3.2. Let a subset of basic contrasts of a block design be represented by the eigenvectors $\mathbf{s}_{\beta 1}, \mathbf{s}_{\beta 2}, ..., \mathbf{s}_{\beta \rho_\beta}$ of $\mathbf{C}$ with respect to $\mathbf{r}^\delta$ corresponding to a common eigenvalue $\varepsilon_\beta > 0$. Then for a set of contrasts $\mathbf{U}'_\beta \boldsymbol{\tau} = \mathbf{A}'_\beta \mathbf{S}'_\beta \mathbf{r}^\delta \boldsymbol{\tau}$, where $\mathbf{S}_\beta = [\mathbf{s}_{\beta 1} : \mathbf{s}_{\beta 2} : \ldots : \mathbf{s}_{\beta \rho_\beta}]$ and $\mathbf{A}_\beta$ is some matrix of ρ_β rows, the intra-block analysis provides the BLUEs of the form

$$(\widehat{\mathbf{U}'_\beta \boldsymbol{\tau}})_{\text{intra}} = \varepsilon_\beta^{-1} \mathbf{A}'_\beta \mathbf{S}'_\beta \mathbf{Q}_1 , \tag{3.11}$$

with the dispersion matrix of the form

$$\text{Cov}[(\widehat{\mathbf{U}'_\beta \boldsymbol{\tau}})_{\text{intra}}] = \varepsilon_\beta^{-1} \mathbf{A}'_\beta \mathbf{A}_\beta \sigma_1^2 \qquad (\sigma_1^2 = \sigma_2^2 + \sigma_e^2) , \tag{3.12}$$

ε_β being the common efficiency factor of the design for all contrasts in the set, $\beta = 0,1,2,...,m-1$.

Proof. This result follows immediately from Theorem 3.2 and Remark 3.1(b).□

Formulae (3.11) and (3.12) of Corollary 3.2 show that the intra-block estimation of contrasts belonging to a subspace spanned by basic contrasts for which the design gives the same efficiency factor is very simple, and that also the structure of the resulting covariance matrix of the obtained BLUEs is simple, everything being controlled by the common efficiency factor. Thus, if the experimental problem has its reflection in distinguishing certain subsets of contrasts, ordered according to their importance, the experiment should be designed in such a way that all members of a specified subset receive a common efficiency factor, of the higher value the more important the contrasts of the subset are. If possible, the design should allow to estimate the most important subset of contrasts with full efficiency (i.e. of value 1).

In addition to Corollary 3.2 it may be noted that due to the decomposition (3.10), a possible g-inverse of the matrix $\mathbf{C}$, i.e. a possible choice of $\mathbf{C}^-$, is

$$\sum_{\beta=0}^{m-1} \varepsilon_\beta^{-1} \mathbf{H}_\beta .$$

This implies that the intra-block treatment sum of squares (see Caliński and Kageyama, 1991, p.108) can be decomposed in the form

$$\mathbf{Q}_1'\mathbf{C}^-\mathbf{Q}_1 = \sum_{\beta=0}^{m-1} \varepsilon_\beta^{-1}\mathbf{Q}_1'\mathbf{H}_\beta\mathbf{Q}_1 ,$$

the βth component, with ρ_β d.f., being related to the βth subset of basic contrasts, for which the common efficiency factor is ε_β (see also Pearce, 1983, p. 75).

All the discussion on basic contrasts conducted till now concerns the estimation within the first stratum, i.e. intra-block, only. This confinement in presenting the theory may be necessary if a general block design, possibly nonproper, is under consideration. In such a general case, as indicated in Section 4 of Caliński and Kageyama (1991), an eigenvalue 1–ε appearing in the inter-block estimation of a contrast may fail in getting a clear interpretation as the efficiency factor in that stratum. It will be shown in the next section that only proper designs induce a block structure which ensures that also the estimation in the inter-block stratum becomes simple.

4. Proper block designs and the orthogonal block structure

As noticed in Section 2, for proper designs some advantageous simplifications occur. These are essential with regard to the inter-block submodel , for which the structure of the covariance matrix, shown in (2.9), is not very satisfactory from the application point of view. In general it obtains a simple form for the extreme case of $K_H\sigma_B^2 = \sigma_U^2$ only, i.e., when the grouping of units into blocks is not successful (see definitions of σ_B^2, σ_U^2 and K_H in Section 2.1 of Caliński and Kageyama, 1991). However, if the design is proper, then Cov($\mathbf{y}_2$) gets a manageable form. Therefore, further development of the theory related to the model (2.1), with the properties (2.2) and (2.3), will be confined here to proper designs, i.e. designs with

$$k_1 = k_2 = \ldots = k_b = k \text{ (say).} \tag{4.1}$$

This will allow the theory to be presented in a unified form.

It can easily be shown that the dispersion matrices in (2.9) and (2.10) are simplified to

$$\text{Cov}(\mathbf{y}_2) = \boldsymbol{\varphi}_2[k\sigma_B^2 + (1-K_H^{-1}k)\,\sigma_U^2 + \sigma_e^2] \tag{4.2}$$

and

$$\text{Cov}(\mathbf{y}_3) = \boldsymbol{\varphi}_3[(1-N_B^{-1}b)\,k\sigma_B^2 + (1-K_H^{-1}k)\,\sigma_U^2 + \sigma_e^2]\,, \tag{4.3}$$

respectively, for any proper block design. Thus, in case of a proper block design the expectation vector and the dispersion matrix for each of the three submodels (2.5) can be written, respectively, as

$$\mathrm{E}(\mathbf{y}_\alpha) = \boldsymbol{\varphi}_\alpha \boldsymbol{\Delta}'\boldsymbol{\tau} \tag{4.4}$$

and

$$\mathrm{Cov}(\mathbf{y}_\alpha) = \boldsymbol{\varphi}_\alpha \sigma_\alpha^2 , \tag{4.5}$$

for $\alpha = 1,2,3$, where, from (2.8), (4.2) and (4.3), the so-called "stratum variances" are

$$\sigma_1^2 = \sigma_U^2 + \sigma_e^2 , \quad \sigma_2^2 = k\sigma_B^2 + (1-K_H^{-1}k)\, \sigma_U^2 + \sigma_e^2$$

and (4.6)

$$\sigma_3^2 = (1-N_B^{-1}b)\, k\sigma_B^2 + (1-K_H^{-1}k)\, \sigma_U^2 + \sigma_e^2 .$$

Furthermore, if $N_B = b$ and $k = K_H$ (the latter implying the equality of all potential block sizes), which can be considered as the most common case, the variances σ_2^2 and σ_3^2 reduce to

$$\sigma_2^2 = k\sigma_B^2 + \sigma_e^2 \quad \text{and} \quad \sigma_3^2 = \sigma_e^2 .$$

On the other hand, under the decomposition (2.4),

$$\mathrm{Cov}(\mathbf{y}) = \sum_{\alpha=1}^{3} \mathrm{Cov}(\mathbf{y}_\alpha) + \sum\sum_{\alpha \neq \alpha'} \mathrm{Cov}(\mathbf{y}_\alpha, \mathbf{y}_{\alpha'}) ,$$

where the covariance matrix of the vectors $\mathbf{y}_\alpha$ and $\mathbf{y}_{\alpha'}$ has the form

$$\mathrm{Cov}(\mathbf{y}_\alpha, \mathbf{y}_{\alpha'}) = \boldsymbol{\varphi}_\alpha \mathrm{Cov}(\mathbf{y}) \boldsymbol{\varphi}_{\alpha'} .$$

It can easily be checked that, under (4.1),

$$\boldsymbol{\varphi}_\alpha \mathrm{Cov}(\mathbf{y}) \boldsymbol{\varphi}_{\alpha'} = \mathbf{0} \tag{4.7}$$

for any $\alpha \neq \alpha'$. [In fact, the assumption (4.1) is necessary for the pair $\alpha=2$, $\alpha'=3$ only.] Thus, for any proper design, the decomposition (2.4) implies not only that

$$\mathrm{E}(\mathbf{y}) = \mathrm{E}(\mathbf{y}_1) + \mathrm{E}(\mathbf{y}_2) + \mathrm{E}(\mathbf{y}_3) = \boldsymbol{\varphi}_1 \boldsymbol{\Delta}'\boldsymbol{\tau} + \boldsymbol{\varphi}_2 \boldsymbol{\Delta}'\boldsymbol{\tau} + \boldsymbol{\varphi}_3 \boldsymbol{\Delta}'\boldsymbol{\tau} ,$$

but also that

$$\mathrm{Cov}(\mathbf{y}) = \mathrm{Cov}(\mathbf{y}_1) + \mathrm{Cov}(\mathbf{y}_2) + \mathrm{Cov}(\mathbf{y}_3) = \boldsymbol{\varphi}_1 \sigma_1^2 + \boldsymbol{\varphi}_2 \sigma_2^2 + \boldsymbol{\varphi}_3 \sigma_3^2 , \tag{4.8}$$

where the matrices φ_1, φ_2 and φ_3 satisfy the conditions (2.6).

The representation (4.8) is a very desirable property, as originally indicated for a more general class of designs by Nelder (1965a). After him, the following definition will be adopted (see also Houtman and Speed, 1983, Section 2.2).

Definition 4.1. An experiment is said to have the orthogonal block structure (OBS) if the dispersion matrix of the random variables observed on the experimental units (plots) has a representation of the form (4.8), where the matrices $\{\varphi_\alpha\}$ are symmetric, idempotent and pairwise orthogonal, summing up to the identity matrix, as in (2.6).

It can now be said that any experiment in a proper block design has the orthogonal block structure, or that it has the OBS property. A natural question, arising immediately, is whether the proper designs are the only block designs inducing the OBS property. The answer is as follows.

Lemma 4.1. An experiment in a block design has under (2.1) the orthogonal block structure if and only if the design is proper.

Proof. Since, from (2.3), (2.6) and (2.7), in general

$$\varphi_1 \mathrm{Cov}(\mathbf{y})\varphi_\alpha = \mathbf{0} \quad \text{for} \quad \alpha = 2,3\,,$$

the representation (4.8) holds if and only if

$$\varphi_2 \mathrm{Cov}(\mathbf{y})\varphi_3 = n^{-1}(\mathbf{D}'\mathbf{k}\mathbf{1}_n' - n^{-1}\mathbf{k}'\mathbf{k}\mathbf{1}_n\mathbf{1}_n')(\sigma_B^2 - K_H^{-1}\sigma_U^2) = \mathbf{0}\,.$$

But $\mathbf{D}'\mathbf{k}\mathbf{1}_n' = n^{-1}\mathbf{k}'\mathbf{k}\mathbf{1}_n\mathbf{1}_n'$ if and only if $\mathbf{k} = (\mathbf{k}'\mathbf{k}/n)\mathbf{1}_n$, i.e., if and only if (4.1) holds. □

The condition (4.1) is, however, not sufficient to obtain for any $\mathbf{s}$ the BLUE of $\mathbf{s}'\mathbf{r}^\delta\boldsymbol{\tau}$ under the overall model (2.1). For this, the design needs to be not only proper but also orthogonal, i.e. such that the condition $k^{-1}\mathbf{N}\mathbf{N}'\mathbf{r}^{-\delta}\mathbf{N} = \mathbf{N}$ holds. Moreover, if the design is connected, this would mean that it has to satisfy the condition $\mathbf{N} = b^{-1}\mathbf{r}\mathbf{1}'$ (see Remark 2.1 of Caliński and Kageyama, 1991). Thus, if a design is proper but not orthogonal, there is no hope of obtaining the BLUEs for all parametric functions of the type $\mathbf{s}'\mathbf{r}^\delta\boldsymbol{\tau} = \mathbf{c}'\boldsymbol{\tau}$ under the model (2.1). It remains then to seek the estimators within each stratum separately, i.e. under the submodels

$$\mathbf{y}_\alpha = \varphi_\alpha \mathbf{y}\,, \quad \alpha = 1, 2, 3, \tag{4.9}$$

which for any proper design have the properties (4.4) and (4.5), with the matrices φ_1 and φ_2 reduced to

$$\varphi_1 = \mathbf{I}_n - k^{-1}\mathbf{D}'\mathbf{D} \tag{4.10}$$

and

$$\varphi_2 = k^{-1}\mathbf{D}'\mathbf{D} - n^{-1}\mathbf{1}_n\mathbf{1}_n', \tag{4.11}$$

respectively, and with $\varphi_3 = n^{-1}\mathbf{1}_n\mathbf{1}_n'$, as defined in Section 2.

Theorem 4.1. If the block design is proper, then under (4.9) a function $\mathbf{w}'\mathbf{y}_\alpha = \mathbf{w}'\varphi_\alpha\mathbf{y}$ is uniformly the BLUE of $\mathbf{c}'\boldsymbol{\tau}$ if and only if $\varphi_\alpha\mathbf{w} = \varphi_\alpha\boldsymbol{\Delta}'\mathbf{s}$, where the vectors $\mathbf{c}$ and $\mathbf{s}$ are in the relation $\mathbf{c} = \boldsymbol{\Delta}\varphi_\alpha\boldsymbol{\Delta}'\mathbf{s}$.

Proof. The proof is exactly as that of Theorem 3.1 of Caliński and Kageyama (1991), on account of (4.4) and (4.5). □

Remark 4.1. Since $\mathbf{1}_v'\boldsymbol{\Delta}\varphi_\alpha = \mathbf{0}$ for $\alpha = 1,2$, the only parametric functions for which the BLUEs may exist under (4.9) with $\alpha = 1,2$ are contrasts. On the other hand, since $\boldsymbol{\Delta}\varphi_3\boldsymbol{\Delta}'\mathbf{s} = n^{-1}(\mathbf{r}'\mathbf{s})\mathbf{r}$ for any $\mathbf{s}$, no contrast will obtain a BLUE under (4.9) for $\alpha = 3$. In fact, a function for which the BLUE exists within the 3rd stratum is $\mathbf{c}'\boldsymbol{\tau} = (\mathbf{s}'\mathbf{r})n^{-1}\mathbf{r}'\boldsymbol{\tau}$, i.e. the overall total or any function proportional to that, the overall mean $n^{-1}\mathbf{r}'\boldsymbol{\tau}$ in particular. [See Remark 3.1, Corollary 3.1(b) and Remark 3.6(a) of Caliński and Kageyama (1991).]

It follows from Theorem 3.5.1 that if for a given $\mathbf{c}$ ($\neq\mathbf{0}$) there exists a vector $\mathbf{s}$ such that $\mathbf{c} = \boldsymbol{\Delta}\varphi_\alpha\boldsymbol{\Delta}'\mathbf{s}$, then the BLUE of $\mathbf{c}'\boldsymbol{\tau}$ in stratum α is obtainable as

$$\widehat{\mathbf{c}'\boldsymbol{\tau}} = \mathbf{s}'\boldsymbol{\Delta}\mathbf{y}_\alpha, \tag{4.12}$$

with the variance of the form

$$\mathrm{Var}(\widehat{\mathbf{c}'\boldsymbol{\tau}}) = \mathbf{s}'\boldsymbol{\Delta}\varphi_\alpha\boldsymbol{\Delta}'\mathbf{s}\sigma_\alpha^2 = \mathbf{c}'(\boldsymbol{\Delta}\varphi_\alpha\boldsymbol{\Delta}')^-\mathbf{c}\,\sigma_\alpha^2, \tag{4.13}$$

where σ_α^2 is the appropriate stratum variance defined in (4.6), and $(\boldsymbol{\Delta}\varphi_\alpha\boldsymbol{\Delta}')^-$ is any g-inverse of $\boldsymbol{\Delta}\varphi_\alpha\boldsymbol{\Delta}'$.

Explicitly, the matrices $\boldsymbol{\Delta}\varphi_\alpha\boldsymbol{\Delta}'$ in (4.13) are:

$$\boldsymbol{\Delta}\varphi_1\boldsymbol{\Delta}' = \mathbf{r}^\delta - k^{-1}\mathbf{N}\mathbf{N}' = \mathbf{C} \quad \text{(the C-matrix)}, \tag{4.14}$$

$$\boldsymbol{\Delta}\varphi_2\boldsymbol{\Delta}' = k^{-1}\mathbf{N}\mathbf{N}' - n^{-1}\mathbf{r}\mathbf{r}' = \mathbf{C}_2 \quad (\mathbf{C}_0 \text{ in Pearce, 1983, p.111}) \tag{4.15}$$

and

$$\boldsymbol{\Delta}\varphi_3\boldsymbol{\Delta}' = n^{-1}\mathbf{r}\mathbf{r}'.$$

Now, returning to the decomposition (2.4), it implies that any function $\mathbf{s}'\boldsymbol{\Delta}\mathbf{y}$, estimating unbiasedly the parametric function $\mathbf{s}'\mathbf{r}^\delta\boldsymbol{\tau} = \mathbf{c}'\boldsymbol{\tau}$, can be resolved into three components, in the form

$$\mathbf{s}'\boldsymbol{\Delta}\mathbf{y} = \mathbf{s}'\mathbf{Q}_1 + \mathbf{s}'\mathbf{Q}_2 + \mathbf{s}'\mathbf{Q}_3\,, \tag{4.16}$$

where $\mathbf{Q}_\alpha = \boldsymbol{\Delta}\mathbf{y}_\alpha = \boldsymbol{\Delta}\boldsymbol{\varphi}_\alpha\mathbf{y}$ ($\alpha = 1,2,3$), i.e., each of the components is a contribution to the estimate from a different stratum.

As stated in Remark 4.1, the only parametric functions for which the BLUEs exist under the submodels $\mathbf{y}_1 = \boldsymbol{\varphi}_1\mathbf{y}$ and $\mathbf{y}_2 = \boldsymbol{\varphi}_2\mathbf{y}$ are contrasts. As will be shown, certain contrasts may admit the BLUEs exclusively under one of these submodels, i.e. either in the intra-block analysis (within the 1st stratum) or in the inter-block analysis (within the 2nd stratum). For other contrasts the BLUEs may be obtained under both of these submodels, i.e. in both of the analyses.

It has been indicated in Lemma 4.1 of Caliński and Kageyama (1991), that a necessary and sufficient condition for the intra-block and inter-block components to estimate the same contrast $\mathbf{c}'\boldsymbol{\tau} = \mathbf{s}'\mathbf{r}^\delta\boldsymbol{\tau}$ (with the accuracy to a constant factor) is that $\mathbf{s}$ is an eigenvector of $\mathbf{C} = \boldsymbol{\Delta}\boldsymbol{\varphi}_1\boldsymbol{\Delta}'$ with respect to $\mathbf{r}^\delta$. This, in particular, remains true for the components $\mathbf{s}'\mathbf{Q}_1 = \mathbf{s}'\boldsymbol{\Delta}\mathbf{y}_1$ and $\mathbf{s}'\mathbf{Q}_2 = \mathbf{s}'\boldsymbol{\Delta}\mathbf{y}_2$ of the resolution (4.16) in case of a proper design. So the necessary and sufficient condition for $E(\mathbf{s}'\mathbf{Q}_1) = \kappa E(\mathbf{s}'\mathbf{Q}_2)$, when $\mathbf{s}'\mathbf{r} = 0$, is

$$\boldsymbol{\Delta}\boldsymbol{\varphi}_1\boldsymbol{\Delta}'\mathbf{s} = \varepsilon\mathbf{r}^\delta\mathbf{s}\,,\ \text{with}\ 0<\varepsilon<1 \quad (\varepsilon = \frac{\kappa}{1+\kappa})\,, \tag{4.17}$$

or its equivalent

$$\boldsymbol{\Delta}\boldsymbol{\varphi}_2\boldsymbol{\Delta}'\mathbf{s} = (1-\varepsilon)\mathbf{r}^\delta\mathbf{s}\,,\ \text{with}\ 0<\varepsilon<1\,. \tag{4.18}$$

Note that the equivalence of (4.17) and (4.18) holds for any block design, whether proper or not, with any ε, provided that $\mathbf{s}'\mathbf{r} = 0$.

Comparing (4.17) and (4.18) with the condition of Theorem 4.1, one can write the following.

Lemma 4.2. If the design is proper, then for any $\mathbf{c} = \mathbf{r}^\delta\mathbf{s}$ such that $\mathbf{s}$ satisfies the equivalent eigenvector conditions (4.17) and (4.18), with $0<\varepsilon<1$, the BLUE of the contrast $\mathbf{c}'\boldsymbol{\tau}$ is obtainable in both of the analyses, in the intra-block analysis and in the inter-block analysis.

Proof. On account of Theorem 4.1, if (4.17) is satisfied, then from $E(\mathbf{s}'\mathbf{Q}_1) = \mathbf{s}'\boldsymbol{\Delta}\boldsymbol{\varphi}_1\boldsymbol{\Delta}\boldsymbol{\tau}$ the function $\varepsilon^{-1}\mathbf{s}'\mathbf{Q}_1$ is under $\mathbf{y}_1 = \boldsymbol{\varphi}_1\mathbf{y}$ the BLUE of $\mathbf{c}'\boldsymbol{\tau} = \mathbf{s}'\mathbf{r}^\delta\boldsymbol{\tau}$. Similarly, if (4.18) is satisfied, then from $E(\mathbf{s}'\mathbf{Q}_2) = \mathbf{s}'\boldsymbol{\Delta}\boldsymbol{\varphi}_2\boldsymbol{\Delta}'\boldsymbol{\tau}$ the function $(1-\varepsilon)^{-1}\mathbf{s}'\mathbf{Q}_2$ is under $\mathbf{y}_2 = \boldsymbol{\varphi}_2\mathbf{y}$ the BLUE of $\mathbf{c}'\boldsymbol{\tau} = \mathbf{s}'\mathbf{r}^\delta\boldsymbol{\tau}$. Since $\mathbf{1}_v'\boldsymbol{\Delta}\boldsymbol{\varphi}_1 = \mathbf{0}' = \mathbf{1}_v'\boldsymbol{\Delta}\boldsymbol{\varphi}_2$, it is easy to check that if any of the two conditions (4.17) and (4.18) holds with $0<\varepsilon<1$ so does the other, any of them implying also the equality $\mathbf{1}_v'\mathbf{c} = \mathbf{r}'\mathbf{s} = \mathbf{0}$. □

It should, however, be noticed that since in general the inequality $0 \leq \varepsilon \leq 1$ holds, the cases not considered in Lemma 4.2 are those of $\varepsilon=0$ and $\varepsilon=1$. To these cases the following result applies.

Lemma 4.3. If the design is proper, then for any $\mathbf{c} = \mathbf{r}^{\delta}\mathbf{s}$ such that $\mathbf{s}$ satisfies one of the eigenvector conditions

$$\Delta\varphi_{\alpha}\Delta'\mathbf{s} = \mathbf{r}^{\delta}\mathbf{s}\,, \qquad \alpha = 1,2,3\,, \tag{4.19}$$

the BLUE of the function $\mathbf{c}'\boldsymbol{\tau}$ is obtainable under the overall model (2.1).

Proof. In case of a proper design the condition for a function $\mathbf{s}'\Delta\mathbf{y}$ to be, under (2.1), uniformly the BLUE of $E(\mathbf{s}'\Delta\mathbf{y}) = \mathbf{s}'\mathbf{r}^{\delta}\boldsymbol{\tau} = \mathbf{c}'\boldsymbol{\tau}$ can be written as

$$\mathbf{N}'\mathbf{s} = k^{-1}\mathbf{N}'\mathbf{r}^{-\delta}\mathbf{N}\mathbf{N}'\mathbf{s} \tag{4.20}$$

(see Caliński and Kageyama, 1991, Theorem 2.1). For $\alpha=1$ the condition (4.19) holds if and only of $\mathbf{N}\mathbf{N}'\mathbf{s} = \mathbf{0}$, i.e. if and only if $\mathbf{N}'\mathbf{s} = \mathbf{0}$, and if the latter holds, (4.20) holds automatically. For $\alpha=2$ the condition (4.19), implying $\mathbf{r}'\mathbf{s} = 0$, holds if and only if $k^{-1}\mathbf{N}\mathbf{N}'\mathbf{s} = \mathbf{r}^{\delta}\mathbf{s}$, and the latter implies (4.20). For $\alpha=3$ it can easily be checked that the condition (4.19) is satisfied if and only if $\mathbf{s} \in C(\mathbf{1}_{v})$, i.e., when $\mathbf{s}$ is proportional to the vector $\mathbf{1}_{v}$, and for such $\mathbf{s}$ the equality (4.20) holds automatically. Thus, any of the three conditions (4.19), for $\alpha = 1,2$ and 3, implies (4.20).

□

Now a general result can be given.

Theorem 4.2. In case of a proper design, for any vector $\mathbf{c} = \mathbf{r}^{\delta}\mathbf{s}$ such that $\mathbf{s}$ satisfies the eigenvector condition

$$\Delta\varphi_{\alpha}\Delta'\mathbf{s} = \varepsilon_{\alpha}\mathbf{r}^{\delta}\mathbf{s}\,, \text{ with } 0<\varepsilon_{\alpha}\leq 1 \qquad (\alpha = 1,2,3)\,, \tag{4.21}$$

where $\varepsilon_1=\varepsilon$, $\varepsilon_2=1-\varepsilon$, $\varepsilon_3=1$, the BLUE of the function $\mathbf{c}'\boldsymbol{\tau}$ is obtainable in the analysis within stratum α [for which (4.21) is satisfied], i.e. under the submodel $\mathbf{y}_{\alpha} = \varphi_{\alpha}\mathbf{y}$, where it gets the form

$$(\widehat{\mathbf{c}'\boldsymbol{\tau}})_{\alpha} = \varepsilon_{\alpha}^{-1}\mathbf{s}'\mathbf{Q}_{\alpha} = \varepsilon_{\alpha}^{-1}\mathbf{c}'\mathbf{r}^{-\delta}\mathbf{Q}_{\alpha}\,, \tag{4.22}$$

and its variance is

$$\mathrm{Var}[(\widehat{\mathbf{c}'\boldsymbol{\tau}})_{\alpha}] = \varepsilon_{\alpha}^{-1}\mathbf{s}'\mathbf{r}^{\delta}\mathbf{s}\sigma_{\alpha}^{2} = \varepsilon_{\alpha}^{-1}\mathbf{c}'\mathbf{r}^{-\delta}\mathbf{c}\sigma_{\alpha}^{2}\,. \tag{4.23}$$

If (4.21) is satisfied with $0<\varepsilon_{\alpha}<1$, then two BLUEs of $\mathbf{c}'\boldsymbol{\tau}$ are obtainable, one under the submodel $\mathbf{y}_1 = \varphi_1\mathbf{y}$ and another under $\mathbf{y}_2 = \varphi_2\mathbf{y}$. If (4.21) is satisfied with $\varepsilon_{\alpha}=1$, then the unique BLUE is obtainable within stratum α only, being simultaneously the BLUE under the overall model (2.1).

Proof. The existence of the BLUEs follows from Lemma 4.2 for α = 1,2 with $0<\varepsilon_\alpha<1$, and from Lemma 4.3 for α = 1,2 with $\varepsilon_\alpha=1$ and for $\alpha=3$, where $\varepsilon_\alpha=1$ in any case. Formulae (4.22) and (4.23) follow from the definitions of $\mathbf{Q}_\alpha$ in (4.16) and from the properties (4.4) and (4.5), considered in view of (4.21). □

Remark 4.2. For $\alpha=1$, i.e. for the intra-block analysis, Theorem 4.2 applies to proper as well as to non-proper block designs, as can be seen from Theorem 4.1 of Caliński and Kageyama (1991).

Remark 4.3. Formula (4.23) shows that the variance of the BLUE of $\mathbf{c}'\boldsymbol{\tau}$ obtainable within stratum α is the smaller the larger is the coefficient ε_α, the minimum variance being attained when $\varepsilon_\alpha=1$, i.e., when (4.22) is the BLUE under the overall model (2.1). Thus, for any proper design, ε_α can be interpreted as the efficiency factor of the analysed design for the function $\mathbf{c}'\boldsymbol{\tau}$ when it is estimated in the analysis within stratum α. On the other hand, $1-\varepsilon_\alpha$ can be regarded as the relative loss of information incurred when estimating $\mathbf{c}'\boldsymbol{\tau}$ in the within stratum α analysis.

Remark 4.4. Since $E(\varepsilon_\alpha^{-1}\mathbf{s}'\mathbf{Q}_\alpha) = \mathbf{s}'\mathbf{r}^\delta\boldsymbol{\tau}$ if and only if (4.21) holds, for a function $\mathbf{c}'\boldsymbol{\tau} = \mathbf{s}'\mathbf{r}^\delta\boldsymbol{\tau}$ to obtain the BLUE within stratum α in the form (4.22), the condition (4.21) is not only sufficient but also necessary.

5. Basic contrasts and the notion of general balance

It has been shown in Section 4 that the eigenvector condition (4.17) implies and is implied by a dual condition (4.18) which in a proper design offers the same kind of simplicity to the inter-block BLUE of the related contrast as the former condition does to the intra-block BLUE of the contrast (Theorem 4.2). This shows that in the case of a proper block design the property determining basic contrasts of the design has its desirable effect not only on the estimation in the intra-block analysis but also on that in the inter-block analysis. Thus, the results presented in Section 3 can now be suitably extended to cover the inter-block analysis as well, provided that the attention is confined to proper designs only.

First, the following can be noted.

Remark 5.1. The efficiency factor ε_α in Theorem 4.2 attains the maximum value 1 for $\alpha=1$ if and only if $\mathbf{N}'\mathbf{s} = \mathbf{0}$, and for α = 2 if and only if $\mathbf{N}'\mathbf{s} \neq \mathbf{0}$ but $\varphi_1\boldsymbol{\Delta}'\mathbf{s} = \mathbf{0}$. On the other hand, a block design is called orthogonal if $\varphi_1\boldsymbol{\Delta}'\mathbf{r}^{-\delta}\mathbf{N} = \mathbf{0}$, i.e., if

$$\varphi_1\boldsymbol{\Delta}'\sum_{i=1}^{v-1}\mathbf{s}_1\mathbf{s}_i'\mathbf{N} = \mathbf{0},$$

where $\{\mathbf{s}_i\}$ represent basic contrasts. Thus, if the vector $\mathbf{s}$ representing a contrast $\mathbf{c}'\boldsymbol{\tau} = \mathbf{s}'\mathbf{r}^{\delta}\boldsymbol{\tau}$ satisfies (4.21) with $\varepsilon_\alpha = 1$ for $\alpha = 1$ or $\alpha = 2$, then the design can be called orthogonal for the contrast. If a proper block design is orthogonal for $\mathbf{c}'\boldsymbol{\tau}$, then $(\widehat{\mathbf{c}'\boldsymbol{\tau}})_\alpha$ given by (4.22), with $\varepsilon_\alpha = 1$, is the BLUE under the overall model (2.1), as stated in Theorem 4.2.

A relevant extension of Theorem 3.1 is the following.

Theorem 5.1. Let $\{\mathbf{c}_i'\boldsymbol{\tau} = \mathbf{s}_i'\mathbf{r}^{\delta}\boldsymbol{\tau},\ \ i = 1,2,\ldots,\upsilon-1\}$ be any set of basic contrasts of a proper block design and let $\{\varepsilon_i,\ \ i = 1,2,\ldots,\upsilon-1\}$ be the corresponding eigenvalues of the matrix $\mathbf{C}$ with respect to $\mathbf{r}^{\delta}$. Then the analysis within stratum α (= 1, 2) provides the BLUEs, of the form

$$(\widehat{\mathbf{c}_i'\boldsymbol{\tau}})_\alpha = \varepsilon_{\alpha i}^{-1}\mathbf{s}_i'\mathbf{Q}_\alpha = \varepsilon_{\alpha i}^{-1}\mathbf{c}_i'\,\mathbf{r}^{-\delta}\mathbf{Q}_\alpha \tag{5.1}$$

with the variances

$$\operatorname{Var}[(\widehat{\mathbf{c}_i'\boldsymbol{\tau}})_\alpha] = \varepsilon_{\alpha i}^{-1}\sigma_\alpha^2 \tag{5.2}$$

and the covariances

$$\operatorname{Cov}[(\widehat{\mathbf{c}_i'\boldsymbol{\tau}})_\alpha, (\widehat{\mathbf{c}_{i'}'\boldsymbol{\tau}})_\alpha] = 0 \qquad (i \neq i'), \tag{5.3}$$

for those of the basic contrasts for which the efficiency factors in stratum α, $\varepsilon_{\alpha i} = \varepsilon_i$ if $\alpha = 1$ and $\varepsilon_{\alpha i} = 1-\varepsilon_i$ if $\alpha = 2$, are nonzero (positive). Also the correlations between $(\widehat{\mathbf{c}_i'\boldsymbol{\tau}})_\alpha$ and $(\widehat{\mathbf{c}_{i'}'\boldsymbol{\tau}})_{\alpha'}$ are zero for $\alpha \neq \alpha'$, whether $i = i'$ or $i \neq i'$.

Proof. On account of Theorem 4.2, formulae (5.1) and (5.2) follow immediately from (4.22) and (4.23), respectively, while (5.3) follows from the formula

$$\operatorname{Cov}(\mathbf{Q}_\alpha) = \boldsymbol{\Delta}\boldsymbol{\varphi}_\alpha\boldsymbol{\Delta}'\sigma_\alpha^2, \tag{5.4}$$

holding due to (4.5), and from the equality $\mathbf{s}_i'\boldsymbol{\Delta}\boldsymbol{\varphi}_\alpha\boldsymbol{\Delta}'\mathbf{s}_{i'} = 0$, satisfied by any pair of vectors $\mathbf{s}_i$, $\mathbf{s}_{i'}$ $(i \neq i')$ in accordance with Definition 3.1 and the equivalence between (4.17) and (4.18). The last statement of the theorem follows from (4.7), from which the vectors $\mathbf{Q}_1$ and $\mathbf{Q}_2$ are uncorrelated. □

Next, an extension of Theorem 3.2 is possible.

Theorem 5.2. For any proper block design let the vectors $\mathbf{s}_1,\ldots,\mathbf{s}_\rho$, $\mathbf{s}_{\rho+1},\ldots,\mathbf{s}_h$, $\mathbf{s}_{h+1},\ldots,\mathbf{s}_{\upsilon-1}$ represent basic contrasts ordered as in Section 3. Then a set of contrasts $\mathbf{U}'\boldsymbol{\tau}$ admits the BLUEs in the analysis within stratum α (=1,2) if and only if the matrix $\mathbf{U}$ can be written as $\mathbf{U} = \mathbf{r}^{\delta}\mathbf{S}_{(\alpha)}\mathbf{A}_{(\alpha)}$, where $\mathbf{S}_{(1)} = [\mathbf{s}_1 : \ldots : \mathbf{s}_h]$ and $\mathbf{A}_{(1)} = [\mathbf{a}_{11} : \ldots : \mathbf{a}_{1h}]'$ is some matrix of h rows, while $\mathbf{S}_{(2)} = [\mathbf{s}_{\rho+1} : \ldots : \mathbf{s}_{\upsilon-1}]$ and $\mathbf{A}_{(2)} = [\mathbf{a}_{2,\rho+1} : \ldots : \mathbf{a}_{2,\upsilon-1}]'$ is some matrix of $\upsilon-\rho-1$ rows.

If $\mathbf{U}$ is such, then the BLUEs provided by the analysis within stratum α are of the form

$$(\widehat{\mathbf{U}'\boldsymbol{\tau}})_\alpha = \mathbf{A}'_{(\alpha)}\boldsymbol{\varepsilon}^{-\delta}_{(\alpha)}\,\mathbf{S}'_{(\alpha)}\mathbf{Q}_\alpha \tag{5.5}$$

and their dispersion matrix is of the form

$$\mathrm{Cov}[(\widehat{\mathbf{U}'\boldsymbol{\tau}})_\alpha] = \mathbf{A}'_{(\alpha)}\boldsymbol{\varepsilon}^{-\delta}_{(\alpha)}\,\mathbf{A}_{(\alpha)}\sigma^2_\alpha\,, \tag{5.6}$$

where $\boldsymbol{\varepsilon}^\delta_{(1)} = \mathrm{diag}[\varepsilon_1, \ldots, \varepsilon_h]$ and $\boldsymbol{\varepsilon}^\delta_{(2)} = \mathrm{diag}[1-\varepsilon_{\rho+1}, \ldots, 1-\varepsilon_{v-1}]$.

Proof. The proof follows the same pattern as that of Theorem 3.2, now on account of Theorem 4.1. It is useful to note that the matrix $\boldsymbol{\Delta}\boldsymbol{\varphi}_\alpha\boldsymbol{\Delta}'$ can be written as

$$\boldsymbol{\Delta}\boldsymbol{\varphi}_\alpha\boldsymbol{\Delta}' = \mathbf{r}^\delta\mathbf{S}_{(\alpha)}\boldsymbol{\varepsilon}^\delta_{(\alpha)}\,\mathbf{S}'_{(\alpha)}\mathbf{r}^\delta \qquad \text{for } \alpha = 1,2\,. \tag{5.7}$$

Formula (5.5) is then obtainable by writing $\mathbf{U} = \boldsymbol{\Delta}\boldsymbol{\varphi}_\alpha\boldsymbol{\Delta}'\mathbf{S}_{(\alpha)}\boldsymbol{\varepsilon}^{-\delta}_{(\alpha)}\mathbf{A}_{(\alpha)}$ and applying Theorem 4.1, while (5.6) results directly from (5.4) and (5.7). □

An extension of Corollary 3.1 is the following.

Corollary 5.1. For any proper block design for which the vectors $\mathbf{s}_1, \ldots, \mathbf{s}_\rho$ and $\mathbf{s}_{h+1}, \ldots, \mathbf{s}_{v-1}$ represent basic contrasts, as in Theorem 5.2, the first ρ receiving the unit efficiency factors in the intra-block analysis and the last $v-h-1$ receiving such efficiency factors in the inter-block analysis (i.e. the zero efficiency factors in the intra-block analysis), a set of contrasts $\mathbf{U}'\boldsymbol{\tau}$ admits the BLUEs under the overall model (2.1) if and only if the matrix $\mathbf{U}$ can (possibly after reordering its columns) be written as

$$\mathbf{U} = \mathbf{r}^\delta[\mathbf{S}_0\mathbf{A}_0 \;:\; \mathbf{S}_m\mathbf{A}_m]\,, \tag{5.8}$$

where $\mathbf{S}_0 = [\mathbf{s}_1 : \ldots : \mathbf{s}_\rho]$, $\mathbf{S}_m = [\mathbf{s}_{h+1} : \ldots : \mathbf{s}_{v-1}]$ and $\mathbf{A}_0$, $\mathbf{A}_m$ are some matrices of conformable numbers of rows. The BLUEs are then obtainable in the form

$$\widehat{\mathbf{U}'\boldsymbol{\tau}} = \begin{bmatrix}\mathbf{A}'_0\mathbf{S}'_0\\ \mathbf{A}'_m\mathbf{S}'_m\end{bmatrix}\boldsymbol{\Delta}\mathbf{y} \tag{5.9}$$

and their dispersion matrix in the form

$$\mathrm{Cov}(\widehat{\mathbf{U}'\boldsymbol{\tau}}) = \begin{bmatrix}\mathbf{A}'_0\mathbf{A}_0\,\sigma^2_1 & 0\\ 0 & \mathbf{A}'_m\mathbf{A}_m\sigma^2_2\end{bmatrix}. \tag{5.10}$$

Proof. The first part of the corollary follows directly from Theorem 2.1 of Caliński and Kageyama (1991), as it can be seen when applying the singular value decomposition

$$\mathbf{N} = \mathbf{r}^{\delta} \sum_{i=\rho+1}^{r} (1-\varepsilon_i)^{1/2} \mathbf{s}_i \mathbf{t}_i' \mathbf{k}^{\delta}$$

to the condition (2.13) of that theorem, which for a proper block design becomes then equivalent to the equality

$$\sum_{i=\rho+1}^{v} (1-\varepsilon_i)^{3/2} \mathbf{t}_i \mathbf{s}_i' \mathbf{r}^{\delta} \mathbf{s} = \sum_{i=\rho+1}^{v} (1-\varepsilon_i)^{1/2} \mathbf{t}_i \mathbf{s}_i' \mathbf{r}^{\delta} \mathbf{s} ,$$

and this, since any $\mathbf{s}$ can be written as $\mathbf{s} = \Sigma_{i=1}^{v-1} a_i \mathbf{s}_i$ where $\{a_i\}$ are some scalars, is equivalent to

$$\sum_{i=\rho+1}^{v-1} (1-\varepsilon_i)^{3/2} a_i \mathbf{t}_i = \sum_{i=\rho+1}^{v-1} (1-\varepsilon_i)^{1/2} a_i \mathbf{t}_i ,$$

which in turn holds if and only if ε_i is either 0 or 1 for any i for which $a_i \neq 0$. The second part of the corollary can be drawn from Theorem 5.2, but it follows also directly from Theorem 2.1 of Caliński and Kageyama (1991) and formula (4.8), in view of (5.8). □

Also Remark 3.1 has its extension.

Remark 5.2.(a) In the notation of Corollary 5.1, a proper block design for which $\rho \geq 1$ and/or $v-h \geq 2$ can be called orthogonal for the set of contrasts $\mathbf{U}'\boldsymbol{\tau} = [\mathbf{S}_0\mathbf{A}_0 : \mathbf{S}_m\mathbf{A}_m]'\mathbf{r}^{\delta}\boldsymbol{\tau}$.

(b) It follows from Theorem 5.2 and Corollary 5.1 that the efficiency factor of a proper block design for a contrast $\mathbf{u}'\boldsymbol{\tau} = \mathbf{a}_{\alpha}'\mathbf{S}_{\alpha}'\mathbf{r}^{\delta}\boldsymbol{\tau}$ estimated in the stratum α analysis is of the form

$$\varepsilon[(\widehat{\mathbf{u}'\boldsymbol{\tau}})_{\alpha}] = \mathbf{a}_{(\alpha)}'\mathbf{a}_{(\alpha)} / \mathbf{a}_{(\alpha)}'\boldsymbol{\varepsilon}_{(\alpha)}^{-\delta}\mathbf{a}_{(\alpha)} \quad (\alpha = 1,2) .$$

At this point it will be useful to notice that when multiplicities of the eigenvalues of $\mathbf{C}$ with respect to $\mathbf{r}^{\delta}$ are taken into account, then appropriate spectral decomposition can be given not only for the matrix C, as in (3.10), but also for the matrix $\mathbf{C}_2$ defined in (4.15). It can be written as

$$\mathbf{C}_2 = \boldsymbol{\Delta}\boldsymbol{\varphi}_2\boldsymbol{\Delta}' = \mathbf{r}^{\delta} \sum_{\beta=1}^{m} (1-\varepsilon_{\beta})\mathbf{H}_{\beta}\mathbf{r}^{\delta} , \tag{5.11}$$

where $\mathbf{H}_1,\ldots,\mathbf{H}_{m-1}$ are as in (3.10), and

$$\mathbf{H}_m = \sum_{j=1}^{\rho_m} \mathbf{s}_{mj}\mathbf{s}'_{mj} = \sum_{j=1}^{v-1-h} \mathbf{s}_{h+j}\mathbf{s}'_{h+j} ,$$

with $\mathbf{s}_{mj} = \mathbf{s}_{h+j}$ for $j = 1,2,..., \rho_m = v-1-h$, and where $\varepsilon_m = 0$.

Now, the following extension of Corollary 3.2 can be given

Corollary 5.2. Let a subset of basic contrasts of a proper block design be represented by the eigenvectors $\mathbf{s}_{\beta 1}, \mathbf{s}_{\beta 2},..., \mathbf{s}_{\beta\rho_\beta}$ of $\mathbf{C}$ with respect to $\mathbf{r}^\delta$ corresponding to a common eigenvalue ε_β (not necessarily positive). Then for a set of contrasts $\mathbf{U}'_\beta\boldsymbol{\tau} = \mathbf{A}'_\beta\mathbf{S}'_\beta\mathbf{r}^\delta\boldsymbol{\tau}$, where $\mathbf{S}_\beta = [\mathbf{s}_{\beta 1} : \mathbf{s}_{\beta 2} : ... : \mathbf{s}_{\beta\rho_\beta}]$ and $\mathbf{A}_\beta$ is some matrix of ρ_β rows, the stratum α (=1,2) analysis provides the BLUEs of the form

$$(\widehat{\mathbf{U}'_\beta\boldsymbol{\tau}})_\alpha = \varepsilon_{\alpha\beta}^{-1} \mathbf{A}'_\beta\mathbf{S}'_\beta\mathbf{Q}_\alpha , \tag{5.12}$$

with the dispersion matrix of the form

$$\mathrm{Cov}[(\widehat{\mathbf{U}'_\beta\boldsymbol{\tau}})_\alpha] = \varepsilon_{\alpha\beta}^{-1}\mathbf{A}'_\beta\mathbf{A}_\beta\sigma_\alpha^2 , \tag{5.13}$$

provided that $\varepsilon_{\alpha\beta}>0$, $\varepsilon_{1\beta} = \varepsilon_\beta$ being the common efficiency factor of the design for the contrasts in the intra-block analysis and $\varepsilon_{2\beta} = 1-\varepsilon_\beta$ being such factor for the contrasts in the inter-block analysis.

Proof. This result follows immediately from Theorem 5.2 and Remark 5.2(b).

□

For completeness, it should also be noticed that, in the spirit of Remark 5.1, one can say that any block design is orthogonal for a function of the type $\mathbf{c}'\boldsymbol{\tau} = a\mathbf{s}'_v\mathbf{r}^\delta\boldsymbol{\tau}$, where a is a nonzero scalar and $\mathbf{s}_v = n^{-1/2}\mathbf{1}_v$. Furthermore, also the matrix $\boldsymbol{\Delta}\boldsymbol{\varphi}_3\boldsymbol{\Delta}'$ appearing in Theorem 4.2 can formally be written in its "spectral decomposition", as

$$\boldsymbol{\Delta}\boldsymbol{\varphi}_3\boldsymbol{\Delta}' = \mathbf{r}^\delta\mathbf{s}_v\mathbf{s}'_v\mathbf{r}^\delta \quad (=n^{-1}\mathbf{r}\mathbf{r}') . \tag{5.14}$$

The three representations (3.10), (5.10) and (5.14), together with the general results established in the present section for proper block designs, give rise to the following concept of balance.

Definition 5.1. A proper block design inducing the OBS property defined by $\{\boldsymbol{\varphi}_\alpha\}$ is said to be generally balanced (GB) with respect to a decomposition

$$C(\boldsymbol{\Delta}') = \oplus_\beta C(\boldsymbol{\Delta}'\mathbf{S}_\beta) \tag{5.15}$$

(the symbol $C(\cdot)$ denoting the column space of a matrix argument and $\oplus_\beta$ denoting the direct sum of the subspaces taken over β), if there exist scalars $\{\varepsilon_{\alpha\beta}\}$ such that for all α (=1,2,3)

$$\mathbf{\Delta}\varphi_\alpha\mathbf{\Delta}' = \sum_\beta \varepsilon_{\alpha\beta}\, \mathbf{r}^\delta \mathbf{H}_\beta \mathbf{r}^\delta \tag{5.16}$$

[the sum being taken over all β that appear in (5.15), $\beta = 0, 1, \ldots, m, m+1$], where $\mathbf{H}_\beta = \mathbf{S}_\beta\mathbf{S}'_\beta$, $\mathbf{H}_{m+1} = \mathbf{s}_\upsilon\mathbf{s}'_\upsilon$, and where the matrices $\{\mathbf{S}_\beta\}$ are such that

$$\mathbf{S}'_\beta\mathbf{r}^\delta\mathbf{S}_\beta = \mathbf{I}_{\rho_\beta} \quad \text{for any } \beta \text{ and } \mathbf{S}'_\beta\mathbf{r}^\delta\mathbf{S}_{\beta'} = \mathbf{0} \text{ for } \beta \neq \beta' .$$

It can easily be shown that Definition 5.1 is equivalent to the definition of GB given by Houtman and Speed (1983, Section 4.1) when applied to a proper block design, and so coincides with the notion of general balance introduced by Nelder (1965b).

The following result explains the sense of Definition 5.1, relating it to the notion of basic contrasts and the theory established for them.

Lemma 5.1. A proper block design is GB with respect to the decomposition (5.15) if and only if the matrices $\{\mathbf{S}_\beta\}$ of Definition 5.1 satisfy the conditions

$$\mathbf{\Delta}\varphi_\alpha\mathbf{\Delta}'\mathbf{S}_\beta = \varepsilon_{\alpha\beta}\mathbf{r}^\delta\mathbf{S}_\beta \tag{5.17}$$

for all α and β. (See also Pearce, 1983, p.110.)

Proof. The implication from (5.17) to (5.16) can be shown as for the representations of $\mathbf{\Delta}\varphi_\alpha\mathbf{\Delta}'$ given in (3.10), (5.11) and (5.14), i.e. by noting that $\mathbf{r}^{-\delta} = \Sigma_{\beta=0}^{m+1} \mathbf{S}_\beta\mathbf{S}'_\beta$, with $\mathbf{S}_{m+1} = \mathbf{s}_\upsilon$, which also implies that

$$\mathbf{\Delta}' = \mathbf{\Delta}'\sum_{\beta=0}^{m} \mathbf{S}_\beta\mathbf{S}'_\beta\,\mathbf{r}^\delta + \mathbf{\Delta}'\mathbf{s}_\upsilon\mathbf{s}'_\upsilon\mathbf{r}^\delta . \tag{5.18}$$

The reverse implication is immediate, due to the $\mathbf{r}^\delta$-orthonormality of the columns of $\{\mathbf{S}_\beta\}$, within and between the matrices. ❑

Remark 5.3. It follows from Lemma 5.1, on account of Definition 3.1, that any proper block design is GB with respect to the decomposition

$$C(\mathbf{\Delta}') = C(\mathbf{\Delta}'\mathbf{S}_0) \oplus C(\mathbf{\Delta}'\mathbf{S}_1) \oplus \ldots \oplus C(\mathbf{\Delta}'\mathbf{S}_m) \oplus C(\mathbf{\Delta}'\mathbf{S}_\upsilon) , \tag{5.19}$$

where the matrices $\mathbf{S}_0, \mathbf{S}_1, \ldots, \mathbf{S}_m$ represent basic contrasts of the design, those represented by the columns of $\mathbf{S}_\beta$ receiving in the intra-block analysis a common efficiency factor $\varepsilon_{1\beta} = \varepsilon_\beta$ and in the inter-block analysis a common efficiency factor $\varepsilon_{2\beta} = 1-\varepsilon_\beta$, and where $\mathbf{s}_\upsilon = n^{-1/2}\mathbf{1}_\upsilon$. Thus, for the estimation of contrasts in the subspaces $C(\mathbf{\Delta}'\mathbf{S}_\beta)$, $\beta = 0,1,\ldots,m$, Corollary 5.2, with (5.12) and (5.13), is applicable.

To see a direct correspondence between the results presented by Houtman and Speed (1983) and those exposed here, it has to be noted that their linear subspace $\mathcal{T}$ coincides with the present $C(\boldsymbol{\Delta}')$ and their subspaces $\{\mathcal{T}_\beta\}$ coincide with the present subspaces $\{C(\boldsymbol{\Delta}'\mathbf{S}_\beta)\}$. Accordingly, their orthogonal projectors $\{\mathbf{T}_\beta\}$ can, in the notation of this paper, be written as

$$\mathbf{T}_\beta = \boldsymbol{\Delta}'\mathbf{S}_\beta\mathbf{S}'_\beta\boldsymbol{\Delta} = \boldsymbol{\Delta}'\mathbf{H}_\beta\boldsymbol{\Delta}\,.$$

Certainly, the equality (5.16) above can equivalently be written as

$$\boldsymbol{\Delta}'\mathbf{r}^{-\delta}\boldsymbol{\Delta}\boldsymbol{\varphi}_\alpha\boldsymbol{\Delta}'\mathbf{r}^{-\delta}\boldsymbol{\Delta} = \sum_\beta \varepsilon_{\alpha\beta}\boldsymbol{\Delta}'\mathbf{H}_\beta\boldsymbol{\Delta}\;,$$

which is exactly the condition of Houtman and Speed (1983, Section 4.1) in their definition of GB.

Also, it should be mentioned that the notion of GB stems back to the early work by Jones (1959), who called an experiment balanced for a contrast if the latter satisfied the condition (4.17), and called it balanced for a set of contrasts if they satisfied this condition with the same eigenvalue. Thus, in his terminology, a block design is balanced for each basic contrast separately, but it is also balanced for any subspace of basic contrast corresponding to a distinct eigenvalue. It is, therefore, natural to call a block design GB for all basic contrasts, provided that the eigenvalues can be interpreted in terms of efficiency factors and relative losses of information on contrasts of interest. This is just what is offered by any proper block design if adequately used.

To illustrate the origins and the sense of the concept of GB, it may be interesting to return to one of the examples of Jones (1959, Section 8), that which was also discussed by Caliński (1971, p.292).

Example 5.1. Consider the following 3×2 factorial experiment, with the incidence matrix

$$\begin{array}{cc c cccccc}
\multicolumn{2}{c}{\text{Treatment at}} & & \multicolumn{6}{c}{\text{Block}} \\
\text{level of} & \text{level of} & & \text{I} & \text{II} & \text{III} & \text{IV} & \text{V} & \text{VI} \\
\text{A} & \text{B} & & & & & & & \\
0 & 0 & & 1 & 0 & 1 & 0 & 1 & 1 \\
1 & 0 & & 1 & 1 & 0 & 1 & 1 & 0 \\
2 & 0 & & 0 & 1 & 1 & 1 & 0 & 1 \\
0 & 1 & \mathbf{N} = & 0 & 1 & 0 & 1 & 1 & 1 \\
1 & 1 & & 1 & 1 & 1 & 0 & 0 & 1 \\
2 & 1 & & 1 & 0 & 1 & 1 & 1 & 0
\end{array}\;.$$

It is a proper equireplicate connected block design with the C-matrix

$$\mathbf{C} = 4\mathbf{I}_6 - \frac{1}{4}\mathbf{NN}',$$

where

$$\mathbf{NN}' = \begin{bmatrix} 4 & 2 & 2 & 2 & 3 & 3 \\ 2 & 4 & 2 & 3 & 2 & 3 \\ 2 & 2 & 4 & 3 & 3 & 2 \\ 2 & 3 & 3 & 4 & 2 & 2 \\ 3 & 2 & 3 & 2 & 4 & 2 \\ 3 & 3 & 2 & 2 & 2 & 4 \end{bmatrix}.$$

The six basic contrasts found by Jones can be represented by the vectors

$\mathbf{s}_1 = [\,1, 1, 1, -1, -1\ -1\,]' / \sqrt{24}$ corresponding to $\varepsilon_1 = 1$,

$\mathbf{s}_2 = [1, -1, 0, 1, -1, 0\,]' / \sqrt{16}$ corresponding to $\varepsilon_2 = 15/16$,

$\mathbf{s}_3 = [\,1, 1, -2, 1, 1, -2\,]' / \sqrt{48}$ corresponding to $\varepsilon_3 = 15/16$,

$\mathbf{s}_4 = [\,1, -1, 0, -1, 1, 0\,]' / \sqrt{16}$ corresponding to $\varepsilon_4 = 13/16$,

$\mathbf{s}_5 = [\,1, 1, -2, -1, -1, 2\,]' / \sqrt{48}$ corresponding to $\varepsilon_5 = 13/16$.

Noting that $\mathbf{r}^{\delta} = 4\mathbf{I}_6$, it can easily be checked that the vectors $\{\mathbf{s}_i\}$ are $\mathbf{r}^{\delta}$-orthonormal, i.e., satisfy the conditions

$$\mathbf{s}_i'\mathbf{r}^{\delta}\mathbf{s}_i = 1 \quad \text{and} \quad \mathbf{s}_i'\mathbf{r}^{\delta}\mathbf{s}_{i'} = 0 \quad \text{if} \quad i \neq i', \quad \text{for} \quad i, i' = 1, 2, ..., 5,$$

and, furthermore, that they are eigenvectors of the matrix $\mathbf{C}$ with respect to $\mathbf{r}^{\delta}$, i.e., satisfy the condition $\mathbf{Cs}_i = \varepsilon_i\mathbf{r}^{\delta}\mathbf{s}_i$ for all i, which is equivalent to the condition $\mathbf{Ms}_i = \alpha_i\mathbf{s}_i$ of Jones (1959, p.175), with $\mathbf{M} = \mathbf{r}^{-\delta}\mathbf{Nk}^{-\delta}\mathbf{N}'$ and $\alpha_i = 1-\varepsilon_i$, as well as to the condition $\mathbf{M}_0\mathbf{s}_i = \mu_i\mathbf{s}_i$ of Caliński (1971, p.281), with $\mathbf{M}_0 = \mathbf{M} - \mathbf{1}_v\mathbf{r}'/n$ and $\mu_i = \alpha_i$, as $\mathbf{r}'\mathbf{s}_i = 0$ for all contrasts. Moreover, it should be noted that $\mathbf{s}_1$ represents the B factor contrast, $\mathbf{s}_2$ and $\mathbf{s}_3$ represent the A factor contrasts and $\mathbf{s}_4$ and $\mathbf{s}_5$ represent the interaction contrasts. Thus, in this example, the design is orthogonal for the B factor contrast, estimated in the intra-block analysis with full efficiency, is balanced for the A factor contrasts, estimated with efficiency 15/16 in the intra-block analysis and with efficiency 1/16 in the inter-block analysis, and is also balanced for the interaction contrasts, estimated with efficiency 13/16 and 3/16 intra-block and inter-block, respectively. Referring now to Remark 5.3, these statements can be summarized by saying that the design is GB with respect to the decomposition (5.19), where $\mathbf{S}_0 = \mathbf{s}_1$, $\mathbf{S}_1 = [\mathbf{s}_2, \mathbf{s}_3]$ and $\mathbf{S}_2 = [\mathbf{s}_4, \mathbf{s}_5]$. But note that because of the meaning of the contrasts, represented by the vectors $\{\mathbf{s}_i\}$, in terms of the treatments actually applied in the experiment, it can also be said that the design is GB with respect to the 3×2 factorial structure of the experimental treatments.

It is essential, when using the GB terminology, always to refer to the decomposition (5.19) with respect to which the balance of the design holds. If the decomposition is coherent with the treatment structure specified by the experimental problem, then the GB property makes sense. However, if the decomposition is meaningless from the point of view of the treatment structure of the experiment, then it may be difficult to make any practical use of that concept. This difficulty will become apparent in the next example.

Example 5.2. Consider a 3 × 2 factorial experiment in which the same incidence matrix as in Example 5.1 is used, but now with different application to the experimental treatments, as follows.

Treatment at level of A	Treatment at level of B
0	0
0	1
1	0
1	1
2	0
2	1

Block: I II III IV V VI

$$\mathbf{N} = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 & 0 \end{bmatrix}.$$

The C-matrix of the design remains the same, and has the same eigenvectors $\{\mathbf{s}_i\}$ with respect to $\mathbf{r}^\delta = 4\mathbf{I}_6$. However, now the meaning of the basic contrasts represented by these eigenvectors is different. The vector $\mathbf{s}_1$ represents a contrast between the first three treatments and the remaining ones. Such contrast is of no interest from the point of view of the present factorial structure of the treatments. Similarly, no other of the contrasts represented by $\{\mathbf{s}_i\}$ is interesting under the above structure of the treatments. To obtain contrasts of interest in this experiment one has to transform the vectors $\{\mathbf{s}_i\}$ by the matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

The vectors $\{\mathbf{s}_i^*\}$ so obtained are

$$\mathbf{s}_1^* = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \mathbf{s}_1$$

$$= [\,1, -1, 1, -1, 1, -1\,]' \,/\sqrt{24}\,,$$

$$\mathbf{s}_2^* = [\,1, 1, -1, -1, 0, 0\,]' \,/\sqrt{16}\,,$$

$$\mathbf{s}_3^* = [\,1, 1, 1, 1, -2, -2,\,]' \,/\sqrt{48}\,,$$

$$\mathbf{s}_4^* = [\,1, -1, -1, 1, 0, 0\,]' \,/\sqrt{16}\,,$$

$$\mathbf{s}_5^* = [\,1, -1, 1, -1, -2, 2\,]' \,/\sqrt{48}\,.$$

They are still $\mathbf{r}^\delta$-orthonormal but, unlike the vectors $\{\mathbf{s}_i\}$, they are not eigenvectors of $\mathbf{C}$ with respect to $\mathbf{r}^\delta = 4\mathbf{I}_6$ any more. Thus, the design considered in the present example is not GB with respect to the 3×2 factorial structure of the experimental treatments, as is the case in Example 5.1.

The two examples above show that in designing an experiment it is not sufficient to choose a design suitable for the block structure. The design is then to be properly adopted to the experimental problem, i.e., one has to assign the design treatments to the experimental treatments in such a way that the design becomes GB with respect to contrasts that are essential from the point of view of the experimental questions under study. For further discussion see Pearce (1983, Section 4.8), and for more illustrative examples see Ceranka (1983, Section 7).

6. Concluding remarks

The unified theory presented in this paper reveals the special role played by basic contrasts in defining the general balance of a block design. Since any proper block design is generally balanced, as stated in Remark 5.3 (see also Houtman and Speed, 1983, Section 5.4), the notion of general balance is interesting *only* from the point of view of the decomposition (5.19) with respect to which the balance holds. Therefore, any block design offered for use in an experiment should be evaluated with regard to that decomposition. The experimenter should be informed on the subspaces of basic contrasts appearing in (5.15) or (5.19) and the efficiency factors receivable by them in the intra-block and in the inter-block analysis. This has already been pointed out by Houtman and Speed (1983, p.1075), who write that these subspaces have to be discovered for each new design or class of designs. Referring directly to block designs, and to partially balanced incomplete block designs in particular, they write (p.1082) that although it is generally not difficult to obtain these subspaces (more precisely orthogonal projections on them) "most writers in statistics have not taken this view point", Corsten (1976) being an notable exception. In fact the canonical (sub)spaces considered by Corsten (1976) are equivalent to those of the basic contrasts considered here. It should also be mentioned, at this point, that the role of basic contrasts (called canonical contrasts) in designing and analysing equireplicate and equiblock-sized balanced factorial experiments was already enhanced by Shah (1960).

The knowledge of the basic contrasts or their subspaces for which a design is GB, and of the efficiency factors assigned to them, allows the experimenter to use the design for an experiment in such a way which best corresponds to the experimental problem. In particular, it allows to implement the design so that the contrasts considered as the most important can be estimated with the highest efficiency in the stratum of the smallest variance, which is the intra-block stratum if the grouping of units into blocks is performed successfully.

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Kontrasty bazowe doświadczalnego układu blokowego ze szczególnym odniesieniem do pojęcia zrównoważenia ogólnego

Streszczenie

W pracy przedstawiono jednolitą teorię kontrastów bazowych oraz, w powiązaniu z nią, przypomniano pojęcie ortogonalnej struktury blokowej i pojęcie zrównoważenia ogólnego. Pokazano, że w modelu randomizacyjnym te dwa pojęcia mają zastosowanie jedynie do układów blokowych właściwych, to znaczy układów o blokach jednakowej wielkości (pojemności). W szczególności wskazano na rolę, jaką odgrywają kontrasty bazowe w definiowaniu zrównoważenia ogólnego. Omówiono także sens praktyczny zrównoważenia układu doświadczalnego ze względu na te kontrasty.

Słowa kluczowe: Kontrasty bazowe, zrównoważenie ogólne, analiza międzyblokowa, analiza wewnątrzblokowa, ortogonalna struktura blokowa, model randomizacyjny.